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Journal of Combinatorial Theory, Series A 115 (2008) 1417–1435

Journal of Combinatorial Theory Series A

www.elsevier.com/locate/jcta

A unified elementary approach to the Dyson, Morris, Aomoto, and Forrester constant term identities

Ira M. Gessel [a], Lun Lv [b], Guoce Xin [b], Yue Zhou [b]

[a] *Department of Mathematics, Brandeis University, Waltham, MA 02454-9110, USA*
[b] *Center for Combinatorics, LPMC-TJKLC, Nankai University, Tianjin 300071, PR China*

Received 25 December 2006

Available online 18 April 2008

Communicated by George Andrews

Abstract

We introduce an elementary method to give unified proofs of the Dyson, Morris, and Aomoto identities for constant terms of Laurent polynomials. These identities can be expressed as equalities of polynomials and thus can be proved by verifying them for sufficiently many values, usually at negative integers where they vanish. Our method also proves some special cases of the Forrester conjecture.

Keywords: Dyson conjecture; Morris identity; Constant term identity

1. Introduction

In 1962, Freeman Dyson [6] conjectured the following identity.

Theorem 1.1. *For nonnegative integers $a_0, a_1, \ldots, a_n$,*

$$\operatorname*{CT}_x \prod_{0\leqslant i\neq j\leqslant n} \left(1-\frac{x_i}{x_j}\right)^{a_j} = \frac{(a_0+a_1+\cdots+a_n)!}{a_0!a_1!\cdots a_n!}, \tag{1.1}$$

where CT_x denotes the constant term.

E-mail addresses: gessel@brandeis.edu (I.M. Gessel), lvlun@mail.nankai.edu.cn (L. Lv), gxin@nankai.edu.cn (G. Xin), zhouyue@mail.nankai.edu.cn (Y. Zhou).

0097-3165/$ – see front matter
doi:10.1016/j.jcta.2008.02.006

Dyson's conjecture was proved independently before his paper was published by Gunson [10] and by Wilson [20], and an elegant recursive proof was later found by Good [9].

Similar identities for constant terms of Laurent polynomials expressed as products are of considerable interest, and we shall discuss several of them in this paper.

First is an identity of Morris [17]. For $a, b, k \in \mathbb{N}$ (the nonnegative integers) define

$$H(x_0, x_1, \dots, x_n; a, b, k) := \prod_{l=1}^{n} \left(1 - \frac{x_l}{x_0}\right)^a \left(1 - \frac{x_0}{x_l}\right)^b \prod_{1 \leqslant i \neq j \leqslant n} \left(1 - \frac{x_i}{x_j}\right)^k. \tag{1.2}$$

Morris proved the following result.

Theorem 1.2.

$$\operatorname*{CT}_x H(x_0, x_1, \dots, x_n; a, b, k) = M_n(a, b, k), \tag{1.3}$$

where

$$M_n(a, b, k) := \prod_{l=0}^{n-1} \frac{(a + b + kl)!(k(l+1))!}{(a + kl)!(b + kl)!k!}. \tag{1.4}$$

Since $H(x_0, \dots, x_n; a, b, k)$ is homogeneous of degree 0 in the x_i, setting $x_0 = 1$ in (1.3) gives an equivalent result, which is the form stated by Morris [17].

A generalization of the Morris identity was given by Aomoto [2], who extended Selberg's integral to obtain a formula equivalent to the following constant term identity [12]: Let

$$A_m(x_0, x_1, \dots, x_n; a, b, k) := \prod_{l=1}^{n} \left(1 - \frac{x_l}{x_0}\right)^{\chi(l \leqslant m)} H(x_0, x_1, \dots, x_n; a, b, k),$$

where $\chi(S)$ equals 1 if the statement S is true and 0 otherwise.

Theorem 1.3.

$$\operatorname*{CT}_x A_m(x_0, x_1, \dots, x_n; a, b, k) = \prod_{l=0}^{n-1} \frac{[a + b + kl + \chi(l \geqslant n - m)]!(k(l+1))!}{[a + kl + \chi(l \geqslant n - m)]!(b + kl)!k!}. \tag{1.5}$$

Another generalization was conjectured by Forrester [7]:

Conjecture 1.4. *We have*

$$\begin{aligned} &\operatorname*{CT}_x \prod_{i,j=n_0+1, i \neq j}^{n} \left(1 - \frac{x_i}{x_j}\right) H(x_0, x_1, \dots, x_n; a, b, k) \\ &\quad = M_{n_0}(a, b, k) \prod_{j=0}^{n_1 - 1} \frac{(j+1)((k+1)j + a + b + kn_0)!(kj + k + j + kn_0)!}{k!((k+1)j + a + kn_0)!((k+1)j + b + kn_0)!}, \end{aligned} \tag{1.6}$$

where $n = n_0 + n_1$.

In [7], Forrester proved the special case $a = b = 0$ (for all k, n_0, and n_1) using a formula due to Bressoud and Goulden [4], and the case $k = 1$ (for all a, b, n_0, and n_1). Kaneko [13,15] proved the special cases $n_1 = 2, 3$ and $n_1 = n - 1$. Moreover, Forrester and Baker [3] formulated a q-analog of Conjecture 1.4, which was recently studied by Kaneko [14].

Our objective in this paper is to introduce an elementary method which leads to new proofs of the Dyson, Morris, and Aomoto identities. Moreover, our method can be used to obtain some partial results on Forrester's conjecture.

The idea behind the proofs is the well-known fact that to prove the equality of two polynomials of degree at most d, it is sufficient to prove that they are equal at $d + 1$ points. This approach was used by Dyson [6] to prove the case $n = 3$ of (1.1). Dyson used Dougall's method [5], in which most of the points are obtained by induction, making heavy use of the symmetry of (1.1). This approach does not seem to generalize beyond $n = 4$. In our approach we use the fact that, as a polynomial in a_0, the right side of (1.1) vanishes for $a_0 = -1, -2, \dots, -(a_1 + \cdots + a_n)$, and we show that the same is true of the left side.

The same idea was used by Gessel and Xin [8] in proving a q-analog of Theorem 1.1, which was conjectured by George Andrews [1] in 1975, and first proved by Zeilberger and Bressoud [22] in 1985.

In all of the proofs, it is routine to show that after fixing all but one parameter, the constant term is a polynomial of degree at most d in the remaining parameter, say a, and that the left side agrees with the right side when $a = 0$. The proofs then differ in showing that both sides vanish at d additional points. In (1.3), (1.5) and (1.6), these polynomials may have multiple roots. We use the polynomial approach to prove the cases in which the roots are distinct, and use another argument, based on the form of the constant term as a function of all the parameters (Proposition 2.4), to extend the result to the general case.

2. Polynomials, vanishing coefficients, and a rationality result

In this section we prove several lemmas that will be needed in the proofs of the constant term identities. First, we show in Lemma 2.1 that the constant terms in these identities can be expressed as polynomials. Next, Lemma 2.2 is useful in showing that these polynomials vanish at certain negative integers. Lemma 2.3, which applies Lemma 2.2 to coefficients of the Dyson product, gives Dyson's conjecture and is also needed in the proof of Proposition 2.4, which allows us to deal with polynomials with multiple roots.

2.1. A polynomial characterization

Fundamental to our approach is the following lemma, which shows that the constant terms we study are polynomials.

Lemma 2.1. *Let $a_0, \dots, a_n$ be nonnegative integers, $d := a_1 + \cdots + a_n$, and let $L(x_1, \dots, x_n)$ be a Laurent polynomial independent of a_0. Then for fixed $a_1, \dots, a_n$, the constant term*

$$Q(a_0, a_1, \dots, a_n) := \operatorname*{CT}_x x_0^{k_0} L(x_1, \dots, x_n) \prod_{l=1}^{n} \left(1 - \frac{x_l}{x_0}\right)^{a_0} \left(1 - \frac{x_0}{x_l}\right)^{a_l} \tag{2.1}$$

is a polynomial in a_0 of degree at most $d + k_0$ for any integer $k_0 \geqslant -d$.

Proof. We can rewrite $Q(a_0, a_1, \dots, a_n)$ in the following form:

$$Q(a_0, a_1, \dots, a_n) = (-1)^{\sum_l a_l} \operatorname*{CT}_x \frac{\prod_{l=1}^n (x_0 - x_l)^{a_0+a_l}}{x_0^{na_0-k_0} \prod_{l=1}^n x_l^{a_l}} L(x_1, \dots, x_n).$$

Expanding each $(x_0 - x_l)^{a_0+a_l}$ as $\sum_{i_l=0}^{\infty} (-1)^{i_l} \binom{a_0+a_l}{i_l} x_0^{a_0+a_l-i_l} x_l^{i_l}$, we get

$$\begin{aligned}
&Q(a_0, a_1, \dots, a_n) \\
&\quad = \operatorname*{CT}_x \sum_{i_1,\dots,i_n} (-1)^{\sum_l a_l + i_l} \binom{a_0+a_1}{i_1} \cdots \binom{a_0+a_n}{i_n} x_0^{k_0+\sum(a_l-i_l)} \prod_{l=1}^n x_l^{i_l-a_l} L(x_1, \dots, x_n) \\
&\quad = \sum_{i_1,\dots,i_n} (-1)^{k_0} \binom{a_0+a_1}{i_1} \cdots \binom{a_0+a_n}{i_n} \cdot \operatorname*{CT}_{x_1,\dots,x_n} \prod_{l=1}^n x_l^{i_l-a_l} L(x_1, \dots, x_n), \qquad (2.2)
\end{aligned}$$

where the sum ranges over all nonnegative integers $i_1, \dots, i_n$ such that $i_1 + \cdots + i_n = d + k_0$. To show that the degree of $Q(a_0, a_1, \dots, a_n)$ in a_0 is at most $d + k_0$, it suffices to show that every term has degree in a_0 at most $d + k_0$. This follows from the fact that $\prod_{l=1}^n x_l^{i_l-a_l} L(x_1, \dots, x_n)$ is a Laurent polynomial independent of a_0, and the fact that the degree of $\binom{a_0+a_1}{i_1} \cdots \binom{a_0+a_n}{i_n}$ in a_0 is $d + k_0$, since $\binom{a_0+a_l}{i_l}$ is a polynomial in a_0 of degree i_l. □

Some corollaries of Lemma 2.1 are given in Appendix A.

Since $Q(a_0, a_1, \dots, a_n)$ as defined in (2.1) is a polynomial in a_0, we can extend it to all integers a_0, not just nonnegative integers. It is useful to extend the meaning of the right side of (2.1) so that (2.1) holds for negative integers a_0. Since $(1 - x_l/x_0)^{a_0}$ for $l \geqslant 1$ is not a Laurent polynomial unless a_0 is a nonnegative integer, we must expand it as a Laurent series, but since

$$\left(1 - \frac{x_l}{x_0}\right)^{a_0} = \left(-\frac{x_l}{x_0}\right)^{a_0} \left(1 - \frac{x_0}{x_l}\right)^{a_0}$$

we might conceivably expand this expression either in powers of x_l/x_0 or of x_0/x_l. To make the expansion well-defined, we need to specify the ring in which we work. We recall that for a ring R, the ring $R((x_1, x_2, \dots, x_n))$ of formal Laurent series in $x_1, \dots, x_n$ with coefficients in R is the set of all formal series in these variables in which only finitely many negative powers of x_j appear for each j. Then it is sufficient to work in the ring $\mathbb{C}((x_0))((x_1, \dots, x_n))$ of formal Laurent series in $x_1, \dots, x_n$ with coefficients in $\mathbb{C}((x_0))$. Informally, we may think of x_0 as larger than all the other variables, so that x_l/x_0 is small for $l \geqslant 1$. Thus we have in this ring the expansion

$$(x_0 - x_l)^{a_0} = x_0^{a_0} \left(1 - \frac{x_l}{x_0}\right)^{a_0} = \sum_{i=0}^{\infty} \binom{a_0}{i} x_0^{a_0-i} (-x_l)^i$$

for all integers a_0; the alternative expansion

$$(x_0 - x_l)^{a_0} = (-1)^{a_0} (x_l - x_0)^{a_0} = (-1)^{a_0} \sum_{i=0}^{\infty} \binom{a_0}{i} x_l^{a_0-i} (-x_0)^i$$

is not valid in this ring unless a_0 is a nonnegative integer.

2.2. Vanishing coefficients

Our goal is to evaluate special cases of the polynomial $Q(a_0, a_1, \dots, a_n)$ given by Lemma 2.1 by finding some of their zeroes. The following lemma helps us to accomplish this.

Lemma 2.2. *Let u_{ij} for $1 \leqslant i < j \leqslant n$ be nonnegative integers and let m_l and v_l for $1 \leqslant l \leqslant n$ be integers. If the coefficient of $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$ in*

$$\frac{\prod_{1 \leqslant i<j \leqslant n} (x_i - x_j)^{u_{ij}}}{\prod_{l=1}^{n} (1 - x_l)^{v_l}} \tag{2.3}$$

is nonzero then for some subset T of $[n] := \{1, 2, \dots, n\}$ we have

$$\sum_{\substack{i,j \in T \\ i<j}} u_{ij} \leqslant \sum_{i \in T} v_i - |T| \tag{2.4}$$

and

$$\sum_{1 \leqslant i<j \leqslant n} u_{ij} \geqslant \sum_{\substack{i,j \in T \\ i<j}} u_{ij} + \sum_{i \in \overline{T}} (m_i + v_i), \tag{2.5}$$

where $\overline{T} = [n] \setminus T$.

Proof. Applying the formula

$$(x_i - x_j)^{u_{ij}} = \big((1 - x_j) - (1 - x_i)\big)^{u_{ij}} = \sum_{\alpha_{ij}+\alpha_{ji}=u_{ij}} (-1)^{\alpha_{ij}} \binom{u_{ij}}{\alpha_{ij}} (1 - x_i)^{\alpha_{ij}} (1 - x_j)^{\alpha_{ji}}$$

and expanding, we can write (2.3) as a linear combination of terms of the form

$$\prod_{l=1}^{n} (1 - x_l)^{-v_l} \prod_{1 \leqslant i<j \leqslant n} (1 - x_i)^{\alpha_{ij}} (1 - x_j)^{\alpha_{ji}} = \prod_{l=1}^{n} (1 - x_l)^{-v_l} \prod_{1 \leqslant i \neq j \leqslant n} (1 - x_i)^{\alpha_{ij}}.$$

Now let $\alpha_i := \sum_{j=1}^{n} \alpha_{ij}$, where $\alpha_{ii} = 0$. Then we may write this product as

$$\prod_{i=1}^{n} (1 - x_i)^{\alpha_i - v_i}. \tag{2.6}$$

If the coefficient of $x_1^{m_1} \cdots x_n^{m_n}$ in (2.6) is nonzero, then for each i, either $\alpha_i - v_i < 0$ or $m_i \leqslant \alpha_i - v_i$. So if the coefficient of $x_1^{m_1} \cdots x_n^{m_n}$ in (2.3) is nonzero, there exist nonnegative integers α_{ij} with $\alpha_{ij} + \alpha_{ji} = u_{ij}$ for $i \neq j$ and $\alpha_{ii} = 0$ and a subset T of $[n]$ such that

$$\alpha_i \leqslant v_i - 1, \quad \text{for } i \in T, \tag{2.7}$$

$$\alpha_i \geqslant m_i + v_i, \quad \text{for } i \in \overline{T}, \tag{2.8}$$

where $\alpha_i = \sum_{j=1}^{n} \alpha_{ij}$ and $\overline{T} = [n] \setminus T$.

Then

$$\sum_{i \in T} \alpha_i = \sum_{\substack{i \in T \\ j \in [n]}} \alpha_{ij} \geqslant \sum_{i,j \in T} \alpha_{ij} = \sum_{\substack{i,j \in T \\ i<j}} (\alpha_{ij} + \alpha_{ji}) = \sum_{\substack{i,j \in T \\ i<j}} u_{ij}. \tag{2.9}$$

Similarly,

$$\alpha_1 + \cdots + \alpha_n = \sum_{1\leqslant i<j\leqslant n} u_{ij}. \tag{2.10}$$

Summing (2.7) for $i \in T$ gives

$$\sum_{i\in T} \alpha_i \leqslant \sum_{i\in T} v_i - |T|,$$

so by (2.9),

$$\sum_{\substack{i,j\in T\\ i<j}} u_{ij} \leqslant \sum_{i\in T} v_i - |T|,$$

which is (2.4).

Summing (2.8) for $i \in \overline{T}$ gives

$$\sum_{i\in \overline{T}} \alpha_i \geqslant \sum_{i\in \overline{T}} (m_i + v_i).$$

Thus by (2.10) and (2.9) we have

$$\sum_{1\leqslant i<j\leqslant n} u_{ij} = \sum_{i\in T} \alpha_i + \sum_{i\in \overline{T}} \alpha_i \geqslant \sum_{\substack{i,j\in T\\ i<j}} u_{ij} + \sum_{i\in \overline{T}} (m_i + v_i),$$

and this is (2.5). $\square$

Next, we apply Lemma 2.2 to prove the vanishing of some coefficients related to the Dyson product.

Lemma 2.3. *Let $a_1, \dots, a_n$ be nonnegative integers and let $d := a_1 + \cdots + a_n$. Let $k_0, k_1, \dots, k_n$ be integers and let k be the sum of the positive integers among $k_1, \dots, k_n$. For a subset $T \subseteq [n]$ we define $\sigma(T) := \sum_{i\in T} a_i$, and we set*

$$J := \bigcup_{T\subset [n]} \{\sigma(T) + 1, \sigma(T) + 2, \dots, \sigma(T) + k\},$$

where the union is over proper subsets T of $[n]$. Then for every a_0 with $-a_0 \in [d] \setminus J$, we have

$$\operatorname*{CT}_x x_0^{k_0} x_1^{k_1} \cdots x_n^{k_n} \prod_{0\leqslant i\neq j\leqslant n} \left(1 - \frac{x_i}{x_j}\right)^{a_j} = 0. \tag{2.11}$$

Proof. First we note that this coefficient is well defined for any negative integer a_0, as explained at the end of Section 2.1. Next, since the product in (2.11) is homogeneous of degree 0 in $x_0, x_1, \dots, x_n$, the constant term does not change if we set x_0 equal to 1, as long as $k_0 + \cdots + k_n = 0$. (Otherwise the constant term is 0.) Setting $a_0 = -h$ and simplifying, we need to show that

$$\operatorname*{CT}_x \frac{\prod_{1\leqslant i<j\leqslant n} (x_i - x_j)^{a_i + a_j}}{x_1^{na_1 - k_1} \cdots x_n^{na_n - k_n} \prod_{l=1}^n (1 - x_l)^{h - a_l}} = 0, \quad \text{for } h \in [d] \setminus J. \tag{2.12}$$

We prove the contrapositive: Suppose that $h \in [d]$ but the left side of (2.12) is not 0. We shall show that $h \in J$; i.e., $\sigma(T) < h \leqslant \sigma(T) + k$ for some $T \subset [n]$.

We apply Lemma 2.2 with $u_{ij} = a_i + a_j$, $m_i = na_i - k_i$, and $v_i = h - a_i$. Then for some subset $T \subseteq [n]$ we have

$$\sum_{\substack{i,j \in T \\ i<j}} (a_i + a_j) \leqslant \sum_{i \in T} (h - a_i) - |T| \tag{2.13}$$

and

$$\sum_{1 \leqslant i<j \leqslant n} (a_i + a_j) \geqslant \sum_{\substack{i,j \in T \\ i<j}} (a_i + a_j) + \sum_{i \in \overline{T}} \bigl((n-1)a_i + h - k_i\bigr). \tag{2.14}$$

Let $t = |T|$. Then (2.13) may be written as

$$(t-1) \sum_{i \in T} a_i \leqslant th - t - \sum_{i \in T} a_i,$$

and this implies that for $T \neq \emptyset$,

$$\sum_{i \in T} a_i < h. \tag{2.15}$$

But (2.15) also holds for $T = \emptyset$, since $h \geqslant 1$. We note that by (2.15), $T \neq [n]$, since $h \leqslant d$.

Similarly, (2.14) gives

$$(n-1) \sum_{i=1}^{n} a_i \geqslant (t-1) \sum_{i \in T} a_i + \sum_{i \in \overline{T}} (n-1)a_i + (n-t)h - \sum_{i \in \overline{T}} k_i.$$

Taking all the terms in the a_i to the left side gives

$$(n-t) \sum_{i \in T} a_i \geqslant (n-t)h - \sum_{i \in \overline{T}} k_i$$

so since $T \neq [n]$,

$$h \leqslant \sum_{i \in T} a_i + k. \tag{2.16}$$

Thus by (2.15) and (2.16),

$$\sum_{i \in T} a_i < h \leqslant \sum_{i \in T} a_i + k,$$

which completes the proof. □

Dyson's conjecture is an easy consequence of Lemma 2.3:

Proof of Theorem 1.1. Fix $a_1, \ldots, a_n \in \mathbb{N}$. Denote by $D_L(a_0)$ and $D_R(a_0)$ the left and right sides of (1.1). It is routine to check that

1. both $D_L(a_0)$ and $D_R(a_0)$ are polynomials in a_0 of degree at most d (by Lemma 2.1);
2. $D_L(0) = D_R(0)$ (by induction on n);
3. $D_R(a_0)$ vanishes when $a_0 = -1, -2, \ldots, -d$.

Now apply Lemma 2.3 with $k_0 = k_1 = \cdots = k_n = 0$, so $k = 0$ and $J = \emptyset$. Then $D_L(a_0)$ also vanishes when $a_0 = -1, -2, \ldots, -d$. The theorem then follows since two polynomials of degree at most d are equal if they agree at $d + 1$ distinct points. □

2.3. A rationality result

We denote by $D_n(x; a_0, a_1, \ldots, a_n)$ the Dyson product

$$\prod_{0 \leqslant i \neq j \leqslant n} \left(1 - \frac{x_i}{x_j}\right)^{a_j}.$$

Good [9] used Lagrange interpolation to derive the following recursion in his proof of the Dyson conjecture: for $a_0, a_1, \ldots, a_n \geqslant 1$, we have

$$D_n(x; a_0, a_1, \ldots, a_n) = \sum_{i=0}^{n} D_n(x; a_0, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_n). \tag{2.17}$$

Using this recursion, Sills and Zeilberger [19], Sills [18], and Lv et al. [16] found explicit formulas for some of the other coefficients of the Dyson product. Their results suggest the following proposition, which we will need in our approach to the Morris, Aomoto, and Forrester constant terms.

Proposition 2.4. *For any Laurent polynomial $L(x_0, \ldots, x_n)$ independent of the a_i,*

$$\operatorname*{CT}_x L(x_0, \ldots, x_n) D_n(x; a_0, \ldots, a_n) = R(a_0, \ldots, a_n) \frac{(a_0 + a_1 + \cdots + a_n)!}{a_0! a_1! \cdots a_n!} \tag{2.18}$$

for some rational function $R(a_0, \ldots, a_n)$ of $a_0, \ldots, a_n$.

Proof. We proceed by induction on n. The $n = 0$ case is trivial. Assume the proposition holds for $n - 1$, i.e.,

$$\operatorname*{CT}_x L(x_1, \ldots, x_n) D_{n-1}(x_1, \ldots, x_n; a_1, \ldots, a_n) = R(a_1, \ldots, a_n) \frac{d!}{a_1! \cdots a_n!},$$

where $d := a_1 + \cdots + a_n$, for any Laurent polynomial $L(x_1, \ldots, x_n)$ independent of a_i. By linearity, it is sufficient to show that (2.18) holds when $L(x_0, x_1, \ldots, x_n)$ is a monomial. Define

$$f = f(a_0, a_1, \ldots, a_n) := \operatorname*{CT}_x x_0^{k_0} x_1^{k_1} \cdots x_n^{k_n} D_n(x_0, \ldots, x_n; a_0, a_1, \ldots, a_n).$$

We construct a rational function $R(a_0, a_1, \ldots, a_n)$ so that

$$f(a_0, \ldots, a_n) = R(a_0, a_1, \ldots, a_n) \frac{(a_0 + a_1 + \cdots + a_n)!}{a_0! a_1! \cdots a_n!} \tag{2.19}$$

holds for all nonnegative integers $a_0, a_1, \ldots, a_n$.

First we show that for each nonnegative integer a_0, there is a rational function $R_{a_0}(a_1, \ldots, a_n)$ of $a_1, \ldots, a_n$ such that

$$f(a_0, \ldots, a_n) = R_{a_0}(a_1, \ldots, a_n) \frac{d!}{a_1! \cdots a_n!}. \tag{2.20}$$

By (2.2), with $i_l = a_0 + a_l - j_l$ for $l = 1, \dots, n$ we have

$$f = \sum_{j_1,\dots,j_n} (-1)^{k_0} \binom{a_0 + a_1}{j_1} \cdots \binom{a_0 + a_n}{j_n} \\ \cdot \underset{x_1,\dots,x_n}{\mathrm{CT}} \prod_{l=1}^{n} x_l^{a_0 - j_l + k_l} D_{n-1}(x_1, \dots, x_n; a_1, \dots, a_n),$$

where the sum ranges over all nonnegative integers $j_1, \dots, j_n$ such that $(a_0 + a_1 - j_1) + \cdots + (a_0 + a_n - j_n) = d + k_0$, i.e., $j_1 + \cdots + j_n = na_0 - k_0$.

Therefore, by the induction hypothesis on n,

$$f = \sum_{j_1+\cdots+j_n = na_0 - k_0} (-1)^{k_0} \binom{a_0 + a_1}{j_1} \cdots \binom{a_0 + a_n}{j_n} R(j_1, \dots, j_n; a_1, \dots, a_n) \frac{d!}{a_1! \cdots a_n!},$$

where for each $j_1, \dots, j_n$, $R(j_1, \dots, j_n; a_1, \dots, a_n)$ is a rational function of $a_1, \dots, a_n$ (which also depends on a_0 and $k_1, \dots, k_n$). Then (2.20) holds with

$$R_{a_0}(a_1, \dots, a_n) \\ = \sum_{j_1+\cdots+j_n = na_0 - k_0} (-1)^{k_0} \binom{a_0 + a_1}{j_1} \cdots \binom{a_0 + a_n}{j_n} R(j_1, \dots, j_n; a_1, \dots, a_n).$$

Now let $\beta_1 = \beta_1(a_1, \dots, a_n), \dots, \beta_r = \beta_r(a_1, \dots, a_n)$, where $r = (2^n - 1)k$, be the linear functions of $a_1, \dots, a_n$ of the form $\sigma(T) + j$, for $T \subset [n]$ and $1 \leqslant j \leqslant k$, where k and σ are as in Lemma 2.3. By Lemma 2.1, $f(a_0, a_1, \dots, a_n)$, for fixed $a_1, \dots, a_n$, is a polynomial in a_0 of degree at most $d + k_0$. Moreover, by Lemma 2.3, $f = 0$ for $-a_0 \in [d] \setminus \{\beta_1, \dots, \beta_r\}$. Thus there is a polynomial $p(a_0)$ of degree at most $r + k_0$ (depending on $a_1, \dots, a_n$) such that

$$f(a_0, a_1, \dots, a_n) = \frac{(a_0 + 1)(a_0 + 2) \cdots (a_0 + d)}{(a_0 + \beta_1)(a_0 + \beta_2) \cdots (a_0 + \beta_r)} p(a_0), \tag{2.21}$$

since f vanishes at the zeroes of the numerator factors that are not canceled by the denominator factors. Comparing with (2.20), we obtain

$$f = \frac{(a_0 + d)!}{a_0!(a_0 + \beta_1)(a_0 + \beta_2) \cdots (a_0 + \beta_r)} p(a_0) = R_{a_0}(a_1, \dots, a_n) \frac{d!}{a_1! \cdots a_n!}.$$

It follows that

$$p(a_0) = \frac{(a_0 + \beta_1) \cdots (a_0 + \beta_r) a_0!}{a_1! \cdots a_n! (d + 1) \cdots (d + a_0)} R_{a_0}(a_1, \dots, a_n) = \frac{1}{a_1! \cdots a_n!} \overline{R}_{a_0}(a_1, \dots, a_n)$$

for some rational function $\overline{R}_{a_0}(a_1, \dots, a_n)$ of $a_1, \dots, a_n$.

Applying the Lagrange interpolation formula, we obtain that

$$p(a_0) = \sum_{l=0}^{r+k_0} p(l) \prod_{i=0,\, i \neq l}^{r+k_0} \frac{a_0 - i}{l - i} = \sum_{l=0}^{r+k_0} \frac{1}{a_1! \cdots a_n!} \overline{R}_l(a_1, \dots, a_n) \prod_{i=0,\, i \neq l}^{r+k_0} \frac{a_0 - i}{l - i}.$$

So by (2.21) we get

$$f = \frac{(a_0 + 1)(a_0 + 2) \cdots (a_0 + d)}{(a_0 + \beta_1)(a_0 + \beta_2) \cdots (a_0 + \beta_r)} \sum_{l=0}^{r+k_0} \frac{1}{a_1! \cdots a_n!} \overline{R}_l(a_1, \dots, a_n) \prod_{i=0,\, i \neq l}^{r+k_0} \frac{a_0 - i}{l - i}$$

$$= R(a_0, \dots, a_n)\frac{(a_0 + a_1 + \cdots + a_n)!}{a_0!a_1!\cdots a_n!},$$

where

$$R(a_0, \dots, a_n) = \frac{1}{(a_0 + \beta_1)\cdots(a_0 + \beta_r)} \sum_{l=0}^{r+k_0} \overline{R}_l(a_1, \dots, a_n) \prod_{i=0,\, i\neq l}^{r+k_0} \frac{a_0 - i}{l - i}$$

is a rational function of $a_0, \dots, a_n$. This completes the induction. □

We note that the proof of Proposition 2.4 shows that the denominator of $R(a_0, \dots, a_n)$ is a product of linear polynomials of the form $a_{i_1} + \cdots + a_{i_m} + j$, where j is a positive integer. This is consistent with the explicit formulas of [16,18,19].

3. The Morris constant term identity

The proof of (1.3) is similar to that of (1.1), so we omit some of the details. We denote by $M_n'(a, b, k)$ the left side of (1.3).

Lemma 3.1. *For fixed $a \in \mathbb{Z}$ and $b \in \mathbb{N}$, if $M_n'(a, b, k) = M_n(a, b, k)$ for $k \geqslant b$, then $M_n'(a, b, k) = M_n(a, b, k)$ for all $k \in \mathbb{N}$.*

Proof. For fixed a and b, by taking the constant term in x_0, we can write $M_n'(a, b, k)$ as $\mathrm{CT}_x\, L D_{n-1}(x_1, \dots, x_n; k, k, \dots, k)$, where D_{n-1} and L are as in Section 2. By Proposition 2.4, $M_n'(a, b, k)/M_n'(0, 0, k)$ is a rational function of k. It is straightforward to check that $M_n(a, b, k)/M_n(0, 0, k)$ is also rational in k. Note that $M_n'(0, 0, k) = M_n(0, 0, k)$ follows from the equal parameter case of the Dyson conjecture. Therefore, the hypothesis implies that $M_n'(a, b, k)/M_n'(0, 0, k) = M_n(a, b, k)/M_n(0, 0, k)$ for all k. The lemma then follows. □

Proof of Theorem 1.2. By setting $a_0 = a, a_i = b$ for $i = 1, \dots, n$ in Lemma 2.1, we see that $M_n'(a, b, k)$ is a polynomial in a of degree at most bn for fixed b and k in $\mathbb{N}$. To see that $M_n(a, b, k)$ also has this property, we rewrite (1.4) as

$$M_n(a, b, k) = \prod_{l=0}^{n-1} \frac{(a + kl + 1)(a + kl + 2)\cdots(a + kl + b)(k(l + 1))!}{(b + kl)!k!}. \tag{3.1}$$

Moreover, it is easily seen that $M_n(a, b, k)$ vanishes if a equals one of the following values:

$$\begin{array}{cccc} -1, & -2, & \dots, & -b; \\ -(k+1), & -(k+2), & \dots, & -(k+b); \\ \vdots & \vdots & \vdots & \vdots \\ -\big[(n-1)k+1\big], & -\big[(n-1)k+2\big], & \dots, & -\big[(n-1)k+b\big]. \end{array} \tag{3.2}$$

Note that these values are distinct if $k \geqslant b$.

The theorem will follow from properties of polynomials, as in the proof of Dyson's conjecture, if we can show that for $bn + 1$ distinct values of a, $M_n'(a, b, k) = M_n(a, b, k)$.

Lemma 3.1 reduces the problem to showing that $M_n'(a, b, k) = M_n(a, b, k)$ if $k \geqslant b$. First we show that the equality holds for $a = 0$: From (1.2), we have

$$\operatorname*{CT}_{x_0} H(x_0, x_1, \dots, x_n; 0, b, k) = \prod_{1 \leqslant i \neq j \leqslant n} \left(1 - \frac{x_i}{x_j}\right)^k. \tag{3.3}$$

Thus by the equal parameter case of Dyson's conjecture, $M_n'(0, b, k) = (nk)!/(k!)^n$, which is equal to $M_n(0, b, k)$.

The remaining values are obtained from the following lemma, which completes the proof of Theorem 1.2. □

Lemma 3.2. *For fixed nonnegative integers b and $k \geqslant b$, $\mathrm{CT}_x\, H(x_0, x_1, \dots, x_n; a, b, k)$ vanishes when a equals one of the values in* (3.2)*.*

Proof. We prove the contrapositive: Suppose that $h \in [nk]$ but the constant term of

$$H(x_0, x_1, \dots, x_n; -h, b, k) = (-1)^{k\binom{n}{2}+nb} \frac{\prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)^{2k}}{\prod_{l=1}^n x_l^{(n-1)k+b} \prod_{l=1}^n (1 - x_l)^{h-b}}$$

is not 0. We shall show that $(t-1)k + b < h \leqslant tk$ for some t, i.e., $-h$ is not in (3.2).

We apply Lemma 2.2 with $u_{ij} = 2k$, $m_i = (n-1)k + b$, and $v_i = h - b$. Then for some subset $T \subseteq [n]$ we have

$$\sum_{\substack{i,j \in T\\ i<j}} 2k \leqslant \sum_{i \in T} (h - b) - |T| \tag{3.4}$$

and

$$\sum_{1 \leqslant i < j \leqslant n} 2k \geqslant \sum_{\substack{i,j \in T\\ i<j}} 2k + \sum_{i \in \overline{T}} \big((n-1)k + h\big). \tag{3.5}$$

Let $t = |T|$. Then (3.4) may be written as

$$(t-1)tk \leqslant t(h-b) - t,$$

and this implies that for $t \neq 0$,

$$(t-1)k + b < h. \tag{3.6}$$

But (3.6) also holds for $t = 0$, since $h \geqslant 1 > b - k$.

Similarly, (3.5) gives

$$(n-1)nk \geqslant t(t-1)k + (n-t)\big((n-1)k + h\big),$$

which simplifies to $t(n-t)k \geqslant (n-t)h$, so for $t \neq n$,

$$h \leqslant tk. \tag{3.7}$$

But (3.7) also holds for $t = n$, since $h \leqslant nk$. Thus by (3.6) and (3.7),

$$(t-1)k + b < h \leqslant tk,$$

which completes the proof. □

We note that it is possible to handle the case $k < b$ directly without applying Proposition 2.4 by using the following fact: z_0 is a root of a polynomial $P(z)$ with multiplicity r if and only if

$\frac{d^i}{dz^i}P(z_0)=0$ for $i=0,1,\dots,r-1$. For instance, we can find roots of $M_n'(a,b,k)$ with multiplicity at least 2 by considering the constant term of

$$\begin{aligned}&\frac{\partial}{\partial a}H(x_0,x_1,\dots,x_n;a,b,k)\\&\quad=\sum_{s=1}^{n}\ln\left(1-\frac{x_s}{x_0}\right)\prod_{l=1}^{n}\left(1-\frac{x_l}{x_0}\right)^a\left(1-\frac{x_0}{x_l}\right)^b\prod_{1\leqslant i\neq j\leqslant n}\left(1-\frac{x_i}{x_j}\right)^k.\end{aligned}$$

4. The Aomoto constant term identity

In this section we will prove Aomoto's identity using our elementary approach. First we note that if $m\leqslant 0$ or $m\geqslant n$, then (1.5) reduces to the Morris identity, so we assume here that $1\leqslant m\leqslant n-1$. The proof is similar to that of the Morris identity but is more complicated. We provide only the details of the key points.

In contrast with the Morris identity, it is not easy to show that (1.5) holds when $a=0$. So instead of proving equality at a $bn+1$st point, we show that both sides of (1.5) have the same leading coefficients as polynomials in a.

Proposition 4.1.

1. *Both sides of* (1.5) *are polynomials in a of degree at most bn.*
2. *The left side and the right side of* (1.5) *have the same leading coefficients in a.*
3. *The right side of* (1.5) *vanishes when a equals one of the values in the following table*:

$$\begin{array}{cccc}-1, & -2, & \dots, & -b;\\ -(k+1), & -(k+2), & \dots, & -(k+b);\\ \vdots & \vdots & \vdots & \vdots\\ -\big[(n-m-1)k+1\big], & -\big[(n-m-1)k+2\big], & \dots, & -\big[(n-m-1)k+b\big];\\ \hline -\big[(n-m)k+2\big], & -\big[(n-m)k+3\big], & \dots, & -\big[(n-m)k+b+1\big];\\ \vdots & \vdots & \vdots & \vdots\\ -\big[(n-1)k+2\big], & -\big[(n-1)k+3\big], & \dots, & -\big[(n-1)k+b+1\big].\end{array} \tag{4.1}$$

Proof of Proposition 4.1 *(sketch).* As with the Morris identity, parts 1 and 3 are straightforward. To show part 2, we rewrite the right side of (1.5) as

$$\prod_{l=0}^{n-1}\frac{[a+kl+\chi(l\geqslant n-m)+1]\cdots[a+kl+\chi(l\geqslant n-m)+b](k(l+1))!}{(b+kl)!k!},$$

whose leading coefficient is now clearly

$$\prod_{l=0}^{n-1}\frac{(k(l+1))!}{(b+kl)!k!}. \tag{4.2}$$

On the other hand, a calculation similar to that in Lemma 2.1 shows that the leading coefficient of the left side of (1.5) equals

$$\frac{1}{(nb)!}\operatorname*{CT}_x (x_1+\cdots+x_n)^{nb}\prod_{l=1}^n x_l^{-b}\prod_{1\leqslant i\neq j\leqslant n}\left(1-\frac{x_i}{x_j}\right)^k,$$

which is equal to (4.2) by Corollary A.3. □

As in the proof of the Morris identity, we may assume k to be sufficiently large by Proposition 2.4. Then we can complete the proof of the Aomoto identity by the following lemma.

Lemma 4.2. *For fixed nonnegative integers b, $k \geqslant b$, and $m \in [n]$, if a equals one of the values in* (4.1)*, then* $\mathrm{CT}_x\, A_m(x_0, x_1, \dots, x_n; a, b, k)$ *vanishes.*

Proof. We prove the contrapositive: Suppose that $h \in [nk+1]$ but the constant term of

$$A_m(x_0,x_1,\dots,x_n;-h,b,k)=(-1)^{k\binom{n}{2}+nb}\frac{\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)^{2k}}{\prod_{l=1}^n x_l^{(n-1)k+b}\prod_{l=1}^n(1-x_l)^{h-b-\chi(l\leqslant m)}}$$

is not equal to 0. We shall show that $(t-1)k+b+1\leqslant h\leqslant tk$ for some t with $1\leqslant t<n-m$, or $(t-1)k+b+1\leqslant h\leqslant tk+1$ for $t=n-m$, or $(t-1)k+b+2\leqslant h\leqslant tk+1$ for some t with $n-m\leqslant t\leqslant n$. That is, $-h$ is not in (4.1).

We apply Lemma 2.2 with $u_{ij}=2k$, $m_i=(n-1)k+b$, and $v_i=h-b-\chi(i\leqslant m)$. Then for some subset $T\subseteq[n]$ we have

$$\sum_{\substack{i,j\in T\\ i<j}}2k\leqslant\sum_{i\in T}\bigl(h-b-\chi(i\leqslant m)\bigr)-|T| \tag{4.3}$$

and

$$\sum_{1\leqslant i<j\leqslant n}2k\geqslant\sum_{\substack{i,j\in T\\ i<j}}2k+\sum_{i\in\overline{T}}\bigl((n-1)k+h-\chi(i\leqslant m)\bigr). \tag{4.4}$$

Let $t=|T|$. Then (4.3) may be written as

$$(t-1)tk\leqslant t(h-b)-t-\sum_{i\in T}\chi(i\leqslant m),$$

and this implies that for $t\neq 0$,

$$(t-1)k+b+2-\chi\bigl(T\cap[m]=\emptyset\bigr)\leqslant h. \tag{4.5}$$

But (4.5) also holds for $t=0$, since $h\geqslant 1\geqslant b-k+1$.

Similarly, (4.4) gives

$$(n-1)nk\geqslant t(t-1)k+(n-t)\bigl((n-1)k+h\bigr)-\sum_{i\in\overline{T}}\chi(i\leqslant m).$$

Taking all terms in the k to the left gives

$$t(n-t)k\geqslant(n-t)h-\sum_{i\in\overline{T}}\chi(i\leqslant m),$$

so for $t\neq n$,

$$h\leqslant tk+\chi\bigl(\overline{T}\subseteq[m]\bigr). \tag{4.6}$$

But (4.6) also holds for $t = n$, since $h \leqslant nk + 1$. Thus by (4.5) and (4.6),

$$(t-1)k + b + 2 - \chi\big(T \cap [m] = \emptyset\big) \leqslant h \leqslant tk + \chi\big(\overline{T} \subseteq [m]\big).$$

Now according to the three cases $t < n - m$, $t = n - m$, and $t > n - m$, the minimum values of $-\chi(T \cap [m] = \emptyset)$ are -1, -1, and 0, respectively, and the maximum values of $\chi(\overline{T} \subseteq [m])$ are 0, 1, and 1, respectively. This completes the proof. □

5. On the Forrester conjecture

We can apply our method to Forrester's constant term to obtain some partial results. It is routine to obtain the following.

Proposition 5.1.

1. *Both sides of* (1.6) *are polynomials in a of degree at most bn.*
2. *If $a = 0$, then the left side of* (1.6) *is equal to the right side of* (1.6).
3. *The right side of* (1.6) *vanishes when a equals one of the values in the following table*:

$$\begin{array}{cccc}
-1, & -2, & \dots, & -b; \\
-(k+1), & -(k+2), & \dots, & -(k+b); \\
\vdots & \vdots & \vdots & \vdots \\
-\big[(n_0-1)k+1\big], & -\big[(n_0-1)k+2\big], & \dots, & -\big[(n_0-1)k+b\big]; \\
\hline
-(n_0k+1), & -(n_0k+2), & \dots, & -(n_0k+b); \\
-\big[(n_0+1)k+2\big], & -\big[(n_0+1)k+3\big], & \dots, & -\big[(n_0+1)k+b+1\big]; \\
\vdots & \vdots & \vdots & \vdots \\
-\big[(n-1)k+n_1\big], & -\big[(n-1)k+n_1+1\big], & \dots, & -\big[(n-1)k+n_1+b-1\big].
\end{array} \tag{5.1}$$

Note that the values in (5.1) are distinct if $k \geqslant b$.

Therefore, by applying Proposition 2.4, Forrester's conjecture would be established if we could show that for sufficiently large k, the left side of (1.6) vanishes when a equals any value in (5.1). However, we are only able to show that it vanishes for some of these values. Denote by $F_{n_0}(x; a, b, k)$ the left side of (1.6). We obtain the following.

Lemma 5.2. *Assume k is sufficiently large. For t with $0 \leqslant t \leqslant n-1$, let $M := \min\{n_1, t\}$. If $a = -h$ with h satisfying the conditions*

$$tk + C_1 + 1 \leqslant h \leqslant tk + b \quad \textit{if } 0 \leqslant t \leqslant n_0, \tag{5.2}$$

$$tk + C_2 + 1 \leqslant h \leqslant tk + b + C_3 \quad \textit{if } n_0 + 1 \leqslant t \leqslant n-1, \tag{5.3}$$

where

$$C_1 = \begin{cases} \lfloor \frac{n_1^2}{4(n-t)} \rfloor & \textit{if } \frac{n_1}{2} \leqslant M, \\ \lfloor \frac{M(n_1-M)}{n-t} \rfloor & \textit{if } \frac{n_1}{2} > M, \end{cases}$$

$$C_2 = \begin{cases} \lfloor \frac{M(n_1-M)}{n-t} \rfloor & \textit{if } \frac{n_1}{2} \geqslant M, \\ \lfloor \frac{n_1^2}{4(n-t)} \rfloor & \textit{if } t - n_0 < \frac{n_1}{2} < M, \\ t - n_0 & \textit{if } \frac{n_1}{2} \leqslant t - n_0, \end{cases}$$
$$C_3 = \left\lceil \frac{(t-n_0+1)(t-n_0)}{t+1} \right\rceil,$$

then $F_{n_0}(x; a, b, k)$ *vanishes.*

Proof. We prove the contrapositive: Suppose that $h \in [nk + n_1]$ but the constant term of

$$F_{n_0}(x; -h, b, k) = \pm \frac{\prod_{1 \leqslant i<j \leqslant n} (x_i - x_j)^{2(k+\chi_{ij}^{n_0})}}{\prod_{l=1}^n x_l^{(n-1)k+b+(n_1-1)\chi(l>n_0)} \prod_{l=1}^n (1-x_l)^{h-b}},$$

where $\chi_{ij}^{n_0} = \chi(i > n_0)\chi(j > n_0)$, is not equal to 0. We shall obtain conditions on h from which the lemma follows.

We apply Lemma 2.2 with $u_{ij} = 2(k + \chi_{ij}^{n_0})$, $m_i = (n-1)k + b + (n_1-1)\chi(i > n_0)$, and $v_i = h - b$. Then for some subset $T \subseteq [n]$ we have

$$\sum_{\substack{i,j\in T\\ i<j}} \left(2k + 2\chi_{ij}^{n_0}\right) \leqslant \sum_{i \in T} (h-b) - |T| \tag{5.4}$$

and

$$\sum_{1 \leqslant i<j \leqslant n} \left(2k + 2\chi_{ij}^{n_0}\right) \geqslant \sum_{\substack{i,j\in T\\ i<j}} \left(2k + 2\chi_{ij}^{n_0}\right) + \sum_{i \in \overline{T}} \big((n-1)k + h + (n_1-1)\chi(i > n_0)\big). \tag{5.5}$$

Let $t = |T|$ and assume that in T there are t_0 elements less than or equal to n_0 and t_1 elements greater than n_0. Then (5.4) may be written as

$$(t-1)tk + t_1(t_1-1) \leqslant t(h-b) - t,$$

where $\max\{0, t-n_0\} \leqslant t_1 \leqslant \min\{n_1, t\}$ by its definition, and the above equation implies that for $T \neq \emptyset$,

$$(t-1)k + b + 1 + \frac{t_1(t_1-1)}{t} \leqslant h. \tag{5.6}$$

But (5.6) also holds for $T = \emptyset$ if $t_1(t_1-1)/t$ is taken as -1 when $t = 0$ (hence $t_1 = 0$), since $h \geqslant 1 > b - k$.

Similarly, (5.5) gives

$$\begin{aligned}&(n-1)nk + (n_1-1)n_1 \\ &\quad \geqslant t(t-1)k + t_1(t_1-1) + (n-t)\big((n-1)k + h\big) + (n_1-1)(n_1-t_1).\end{aligned}$$

Taking all terms in the k to the left gives

$$t(n-t)k \geqslant (n-t)h - t_1(n_1-t_1),$$

so for $T \neq [n]$,

$$h \leqslant tk + \frac{t_1(n_1 - t_1)}{n - t}. \tag{5.7}$$

But (5.7) also holds for $T = [n]$ if $t_1(n_1 - t_1)/(n - t)$ is taken as n_1 when $t = n$ (hence $t_1 = n_1$), since $h \leqslant nk + n_1$.

Thus by (5.6) and (5.7),

$$h \in I(t, t_1) := \left[(t-1)k + b + 1 + \frac{t_1(t_1 - 1)}{t}, tk + \frac{t_1(n_1 - t_1)}{n - t} \right].$$

It follows that $F_{n_0}(x; -h, b, k)$ vanishes if

$$h \in [nk + n_1] \Big\backslash \bigcup_{t_1 \leqslant t} I(t, t_1),$$

where t ranges from 0 to n and t_1 ranges from $\max\{0, t - n_0\}$ to $\min\{n_1, t\}$.

The above condition can be simplified further: For $0 \leqslant t \leqslant n - 1$ if

$$tk + \left\lfloor \frac{r_1(n_1 - r_1)}{n - t} \right\rfloor + 1 \leqslant h \leqslant tk + b + \left\lceil \frac{r_2(r_2 - 1)}{t + 1} \right\rceil \tag{5.8}$$

holds for every r_1 with $\max\{0, t - n_0\} \leqslant r_1 \leqslant \min\{n_1, t\}$ and r_2 with $\max\{0, t - n_0 + 1\} \leqslant r_2 \leqslant \min\{n_1, t + 1\}$, then $F_{n_0}(x; -h, b, k)$ vanishes. This is because when k is sufficiently large, the left and right endpoints of $I(t, r_1)$ are always to the left of the corresponding endpoints of $I(t + 1, r_2)$ for any r_1 and r_2 in their range. Therefore, after removing the intervals $\bigcup_{r_1} I(t, r_1)$ and $\bigcup_{r_2} I(t + 1, r_2)$, each remaining value of h belongs to an open interval (possibly empty), from the right endpoint of $I(t, r_1)$ to the left endpoint of $I(t + 1, r_2)$ for some r_1 and r_2.

By analyzing the extreme values of (5.8) among the range of r_1 and r_2, it is straightforward to obtain (5.2) and (5.3). □

Remark 5.3. The first two lines of (5.1) are always implied by Lemma 5.2 to be roots for $n_0 \geqslant 1$. This follows easily by checking the cases $t = 0$ and $t = 1$.

Corollary 5.4. *Conjecture* 1.4 *holds in the extreme cases $n_1 = 2$ and $n_1 = n - 1$.*

Proof. We verify this directly by Lemma 5.2.

If $n_1 = 2$, then $n_0 = n - 2$. The first two lines of (5.1) are roots by Remark 5.3. If $2 \leqslant t \leqslant n_0 = n - 2$, then $M = \min\{2, t\} = 2$, and $C_1 = 0$. Thus we obtain the range $tk + 1 \leqslant h \leqslant tk + b$, which is consistent with the $(t + 1)$st line of (5.1). If $t = n - 1$, then $M = 2$, and $C_2 = C_3 = 1$. Thus we obtain the range $(n - 1)k + 2 \leqslant h \leqslant (n - 1)k + b + 1$, which is consistent with the nth line of (5.1). Therefore, Lemma 5.2 implies that all values of a in (5.1) are roots, and Forrester's conjecture holds in this case.

If $n_1 = n - 1$, then $n_0 = 1$. The cases $t \leqslant n_0 = 1$ of (5.1) are dealt with in Remark 5.3. If $2 \leqslant t \leqslant n - 1$, then $M = t$, $C_3 = \lceil \frac{t(t-1)}{t+1} \rceil = \lceil \frac{(t+1)(t-1)-(t-1)}{t+1} \rceil = t - 1$, and all three cases of C_2 are equal to $t - 1$. Thus we obtain the range $tk + t \leqslant h \leqslant tk + b + t - 1$, which is consistent with that of (5.1). As in the case $n_1 = 2$, Forrester's conjecture holds. □

Proposition 5.5. *Forrester's conjecture holds when $n \leqslant 5$.*

Proof. The cases $n \leqslant 4$ are consequences of Corollary 5.4; the case $n = 5$ can be verified by Lemma 5.2. $\square$

Further routine calculations by Lemma 5.2 gives us the following table:

$$\begin{array}{llllllll} n_1 = & 2 & 3 & 4 & 5 & \cdots & n-3 & n-2 & n-1 \\ M_r = & 0 & 1 & 4 & 8 & \cdots & 2n_1 + 2\lfloor \frac{n_1}{2} \rfloor - 5 & n_1 + \lfloor \frac{n_1}{2} \rfloor - 6 & 0 \end{array} \tag{5.9}$$

where M_r is an upper bound for the number of *missing roots* in (5.1), i.e., roots that are not implied by Lemma 5.2. For brevity, we verify in detail only the case $n_1 = 3$; the other cases are similar.

For $n_1 = 3$, we have $n_0 = n - 3$. The cases $t = 0, 1$ are guaranteed by Remark 5.3, so henceforth we will always assume $t \geqslant 2$ and (by Proposition 5.5) $n \geqslant 6$. If $t = 2$, then $M = 2$ and $\frac{n_1}{2} = 3/2 < M$. Therefore $C_1 = \lfloor \frac{3^2}{4(n-2)} \rfloor = 0$, for $n \geqslant 5$. If $3 \leqslant t \leqslant n-3$, then $M = 3$, and $C_1 = \lfloor \frac{n_1^2}{4(n-t)} \rfloor = \lfloor \frac{9}{4(n-t)} \rfloor = 0$. If $t = n-2$, then $M = 3$ and $t - n_0 = 1$. It follows that $t - n_0 < \frac{n_1}{2} < M$, $C_2 = \lfloor \frac{n_1^2}{4(n-t)} \rfloor = 1$, $C_3 = 1$. If $t = n-1$, then $M = 3$. This implies that $\frac{n_1}{2} \leqslant t - n_0$, $C_2 = t - n_0 = 2$, and $C_3 = 1$.

In conclusion, in the case $n_1 = 3$, only one root $-[(n-1)k + b + 2]$ is not implied by Lemma 5.2.

Corollary 5.6. *Conjecture* 1.4 *holds in the case* $n_1 = 3$.

Proof. As we have just seen, for $n_1 = 3$ we are missing only one root. But by [3], we know that the q-generalization of (1.6) holds when $a = k$. Thus if we let $q \to 1$ in this result, we get $a = k$ as our $(bn+1)$st point. $\square$

We conclude this paper by the following observation. Let us take the Forrester constant term as an example. In the proof of Lemma 2.2, we made the expansion

$$\prod_{1 \leqslant i < j \leqslant n} (y_i - y_j)^{2k} \prod_{n_0+1 \leqslant i < j \leqslant n} (y_i - y_j)^2 = \sum_{\gamma} c_\gamma y_1^{\gamma_1} \cdots y_n^{\gamma_n},$$

where $y_i = 1 - x_i$, and try to show that the constant term associated with $y_1^{\gamma_1} \cdots y_n^{\gamma_n}$ is equal to 0 for each γ. However, it would be sufficient to show that for each γ, either the associated constant term is 0 or $c_\gamma = 0$ (after cancellation). We *conjecture* that in our approach to Forrester's conjecture, $c_\gamma = 0$ when Lemma 2.2 does not apply. We have checked our conjecture for $n \leqslant 6$ and $k \leqslant 3$.

Acknowledgments

The authors would like to thank the referees for helpful suggestions to improve the presentation. Lun Lv and Yue Zhou would like to acknowledge the helpful guidance of their supervisor William Y.C. Chen. The third author was supported by the 973 Project, the PCSIRT project of the Ministry of Education, the Ministry of Science and Technology and the National Science Foundation of China.

Appendix A. Consequences of the polynomial approach

From Lemma 2.1 and its proof, we can deduce the following result.

Corollary A.1. *Let d and Q be as in Lemma 2.1 with $k_0 = 0$. Then the leading coefficient of $Q(a_0, \dots, a_n)$ in a_0 is*

$$\frac{1}{d!} \operatorname*{CT}_x (x_1 + \cdots + x_n)^d \prod_{l=1}^n x_l^{-a_l} L(x_1, \dots, x_n), \tag{A.1}$$

and the second leading coefficient of $Q(a_0, \dots, a_n)$ in a_0 is

$$\operatorname*{CT}_x \left(\sum_{l=1}^n \frac{a_l x_l}{(d-1)!} \left(\sum_{i=1}^n x_i \right)^{d-1} - \frac{1}{2} \sum_{l=1}^n \frac{x_l^2}{(d-2)!} \left(\sum_{i=1}^n x_i \right)^{d-2} \right) \prod_{l=1}^n x_l^{-a_l} L(x_1, \dots, x_n). \tag{A.2}$$

Proof. Taking the leading coefficient of (2.2) in a_0 gives

$$\begin{aligned}
&\sum_{i_1, \dots, i_n} \frac{1}{i_1! i_2! \cdots i_n!} \operatorname*{CT}_x \prod_{l=1}^n x_l^{i_l - a_l} L(x_1, \dots, x_n) \\
&\quad = \frac{1}{d!} \operatorname*{CT}_x \sum_{i_1, \dots, i_n} \frac{d! x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}}{i_1! i_2! \cdots i_n!} \prod_{l=1}^n x_l^{-a_l} L(x_1, \dots, x_n) \\
&\quad = \frac{1}{d!} \operatorname*{CT}_x (x_1 + \cdots + x_n)^d \prod_{l=1}^n x_l^{-a_l} L(x_1, \dots, x_n).
\end{aligned}$$

Taking the second leading coefficient of (2.2) gives

$$\operatorname*{CT}_x \sum_{i_1, \dots, i_n} \sum_{l=1}^n \frac{i_l(2a_l - i_l + 1)}{2} \frac{x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}}{i_1! i_2! \cdots i_n!} \prod_{l=1}^n x_l^{-a_l} L(x_1, \dots, x_n),$$

which can be rewritten as (A.2). □

Applying Corollary A.1 to the Dyson conjecture gives the following identity, which appeared in [21, Corollary 5.4].

Corollary A.2.

$$\operatorname*{CT}_x (x_1 + \cdots + x_n)^{a_1 + \cdots + a_n} \prod_{l=1}^n x_l^{-a_l} \prod_{1 \leqslant i \neq j \leqslant n} \left(1 - \frac{x_i}{x_j}\right)^{a_j} = \frac{(a_1 + \cdots + a_n)!}{a_1! a_2! \cdots a_n!}. \tag{A.3}$$

We omit the formula for the second leading coefficient, which is more complicated.

Applying Corollary A.1 to Morris's identity gives the following result, which is needed for the proof of Aomoto's identity. We remark that (A.4) was also obtained in [11, Proposition 2.2] through a complicated calculation.

Corollary A.3.

$$\operatorname*{CT}_x (x_1+\cdots+x_n)^{nb}\prod_{l=1}^{n} x_l^{-b}\prod_{1\leqslant i\neq j\leqslant n}\left(1-\frac{x_i}{x_j}\right)^k=(nb)!\prod_{l=0}^{n-1}\frac{(k(l+1))!}{(b+kl)!k!}, \tag{A.4}$$

$$\operatorname*{CT}_x \frac{x_1^2(\sum_{i=1}^n x_i)^{nb-2}}{\prod_{l=1}^n x_l^b}\prod_{1\leqslant i\neq j\leqslant n}\left(1-\frac{x_i}{x_j}\right)^k$$
$$=-b(nk-k-b+1)(nb-2)!\prod_{l=0}^{n-1}\frac{(k(l+1))!}{(b+kl)!k!}. \tag{A.5}$$

The proofs of (A.3), (A.4) and (A.5) are straightforward.

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